

On the Nonexistence of Quadratic Lyapunov Functions for Consensus Algorithms

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Abstract

We provide an example proving that there exists no quadratic Lyapunov function for a certain class of linear agreement/consensus algorithms, a fact that had been numerically verified in [6]. We also briefly discuss sufficient conditions for the existence of such a Lyapunov function.

I. INTRODUCTION

We examine a class of algorithms that can be used by a group of agents (e.g., UAVs, nodes of a communication network, etc.) in order to reach consensus on a common opinion (represented by a scalar or vector), starting from different initial opinions, and possibly in the presence of severe restrictions on inter-agent communications.

We focus on a particular algorithm, whereby, at each time step, every agent averages its own opinion with received messages containing the current opinions of some other agents. While this algorithm is known to converge under mild conditions, convergence proofs usually rely on the “span norm” of the vector of opinions. In this note, we address the question of whether convergence can also be established using a quadratic Lyapunov function. Among other reasons, this question is of interest because of its potential implications on convergence time analysis. A negative answer to this question was provided in [6], where the nonexistence of a quadratic Lyapunov function was verified numerically. In this paper, we provide an explicit example and proof of this fact.

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In Section II we give some definitions and formally state the problem. Section III contains the main result and its proof. Section IV provides some additional perspective, together with some conditions under which a quadratic Lyapunov function is guaranteed to exist.

II. THE AGREEMENT ALGORITHM

We consider a set $N = \{1, 2, \dots, n\}$ of agents embedded, at each nonnegative integer time t , in a directed graph $G(t) = (N, \mathcal{E}(t))$. We assume that $(i, i) \in \mathcal{E}(t)$, for all i and t . We define $N_i(t) = \{j \mid (j, i) \in \mathcal{E}(t)\}$, and let $d_i(t)$ be the cardinality of $N_i(t)$.

Each agent i starts with a scalar value $x_i(0)$. At each time t , agent i receives from every agent $j \in N_i(t)$ a message with the value of $x_j(t)$, and uses the received values to perform the update

$$x_i(t+1) = \sum_{j=1}^n a_{ij}(t)x_j(t),$$

where the $a_{ij}(t)$ are nonnegative coefficients that satisfy $a_{ij}(t) = 0$ if $(j, i) \notin \mathcal{E}(t)$, and $\sum_{j \in N_i(t)} a_{ij}(t) = 1$, so that $x_i(t+1)$ is a weighted average of the values $x_j(t)$ held by the agents at time t . We define the vector $x(t) = (x_1(t), \dots, x_n(t))$, and note that the algorithm can be written in the form $x(t+1) = A(t)x(t)$.

We next state some conditions under which the agreement algorithm is guaranteed to converge.

Assumption 1: There exists some $\alpha > 0$ such that if $(j, i) \in \mathcal{E}(t)$, then $a_{ij}(t) \geq \alpha$.

Assumption 2: (Bounded intercommunication intervals) There is some B such that for every nonnegative integer k , the graph $(N, \mathcal{E}(kB) \cup \mathcal{E}(kB+1) \cup \dots \cup \mathcal{E}((k+1)B))$ is strongly connected.

Theorem 3: Under Assumptions 1-2, and for every $x(0)$, the components $x_i(t)$, $i = 1, \dots, n$, converge to a common limit.

Theorem 3 is presented in [11] and is proved in [10] (under a slightly different version of Assumption 2), as well as in [6], for a special case to be considered below; see also [5], [9] for generalizations and extensions. On the other hand, if the graphs $G(t)$ are symmetric, namely, $(i, j) \in \mathcal{E}(t)$ if and only if $(j, i) \in \mathcal{E}(t)$, Assumption 2 can be replaced by the weaker requirement that the graph $(N, \cup_{s \geq t} \mathcal{E}(s))$ is strongly connected for every $t \geq 0$; see [5], [7], [4], [9].

We will focus on a special case, motivated from the model of Vicsek et al. [12], and studied in [6], to be referred to as the *symmetric, equal-neighbor* model. In this model, the graphs $G(t)$ are symmetric, and $a_{ij}(t) = 1/d_i(t)$, for every $(j, i) \in \mathcal{E}(t)$. Thus, each node i forms an unweighted average of the values $x_j(t)$ that it has access to (including its own).

Theorem 1 is usually proved by showing that the “span norm” $\max_i x_i(t) - \min_i x_i(t)$ is guaranteed to decrease after a certain number of iterations. Unfortunately, this proof method usually gives an overly conservative bound on the convergence time of the algorithm. Tighter bounds on the convergence time would have to rely on alternative Lyapunov functions, such as quadratic ones, of the form $x^T M x$, if they exist.

Although quadratic Lyapunov functions can always be found for linear systems, they may fail to exist when the system is allowed to switch between a fixed number of linear modes. On the other hand, there are classes of such switched linear systems that do admit quadratic Lyapunov functions. See [8] for a broad overview of the literature on this subject. For the symmetric, equal-neighbor model this issue was investigated in [6]. The authors write:

“...no such common Lyapunov matrix M exists. While we have not been able to construct a simple analytical example which demonstrates this, we have been able to determine, for example, that no common quadratic Lyapunov function exists for the class of all [graphs which have] 10 vertices and are connected. One can verify that this is so by using semidefinite programming...”

The main contribution of this note is to provide an analytical example that proves this fact.

III. THE EXAMPLE

Let us fix a positive integer n . We start by defining a class \mathcal{Q} of functions with some minimal desired properties of quadratic Lyapunov functions. Let \mathbf{e} be the vector in \mathbb{R}^n with all components equal to 1. A square matrix is said to be *stochastic* if it is nonnegative and the sum of the entries in each row is equal to one. Let $\mathcal{A} \subset \mathbb{R}^{n \times n}$ be the set of stochastic matrices A such that: (i) $a_{ii} > 0$, for all i ; (ii) all positive entries on any given row of A are equal; (iii) $a_{ij} > 0$ if and only if $a_{ji} > 0$; (iv) the graph associated with the set of edges $\{(i, j) \mid a_{ij} > 0\}$ is connected. These are precisely the matrices that correspond to a single iteration of the equal-neighbor algorithm on symmetric, connected graphs.

Definition 4: A function $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to the class \mathcal{Q} if it is of the form $Q(x) = x^T M x$, where:

- (a) The matrix $M \in \mathbb{R}^{n \times n}$ is nonzero, symmetric, and nonnegative definite.
- (b) For every $A \in \mathcal{A}$, and $x \in \mathbb{R}^n$, we have $Q(Ax) \leq Q(x)$.
- (c) We have $Q(\mathbf{e}) = 0$.

Note that condition (b) may be rewritten in matrix form as

$$x^T A^T M A x \leq x^T M x, \quad \text{for all } A \in \mathcal{A}, \text{ and } x \in \mathbb{R}^n. \quad (\text{III.1})$$

The rationale behind condition (c) is as follows. Let S be the subspace spanned by the vector \mathbf{e} . Since we are interested in convergence to the set S , and every element of S is a fixed point of the algorithm, it is natural to require that $Q(\mathbf{e}) = 0$, or, equivalently,

$$M \mathbf{e} = 0.$$

Of course, for a Lyapunov function to be useful, additional properties would be desirable. For example we should require some additional condition that guarantees that $Q(x(t))$ eventually decreases. However, according to Theorem 5, even the minimal requirements in Definition 4 are sufficient to preclude the existence of a quadratic Lyapunov function.

Theorem 5: Suppose that $n \geq 8$. Then, the class \mathcal{Q} (cf. Definition 4) is empty.

The idea of the proof is as follows. Using the fact the dynamics of the system are essentially the same when we rename the components, we show that if $x^T M x$ has the desired properties, so does $x^T Z x$ for a matrix Z that has certain permutation-invariance properties. This leads us to the conclusion that there is essentially a single candidate Lyapunov function, for which a counterexample is easy to develop.

Recall that a permutation of n elements is a bijective mapping $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. Let Σ be the set of all permutations of n elements. For any $\sigma \in \Sigma$, we define a corresponding permutation matrix P_σ by letting the i th component of $P_\sigma x$ be equal to $x_{\sigma(i)}$. Note that $P_\sigma^{-1} = P_\sigma^T$, for all $\sigma \in \Sigma$. Let \mathcal{P} be the set of all permutation matrices corresponding to permutations in Σ .

Lemma 6: Let $M \in \mathcal{Q}$. Define Z as

$$Z = \sum_{P \in \mathcal{P}} P^T M P.$$

Then, $Z \in \mathcal{Q}$.

Proof: For every matrix $A \in \mathcal{A}$, and any $P \in \mathcal{P}$, it is easily seen that $PAP^T \in \mathcal{A}$. This is because the transformation $A \mapsto PAP^T$ amounts to permuting the rows and columns of A , which is the same as permuting (renaming) the nodes of the graph.

We claim that if $M \in \mathcal{Q}$ and $P \in \mathcal{P}$, then $P^T M P \in \mathcal{Q}$. Indeed, if M is nonzero, symmetric, and nonnegative definite, so is $P^T M P$. Furthermore, since $Pe = e$, if $Me = 0$, then $P^T M P e = 0$. To establish condition (b) in Definition 4, let us introduce the notation $Q_P(x) = x^T (P^T M P) x$. Fix a vector $x \in \mathbb{R}^n$, and $A \in \mathcal{A}$; define $B = PAP^T \in \mathcal{A}$. We have

$$\begin{aligned} Q_P(Ax) &= x^T A^T P^T M P A x \\ &= x^T P^T P A^T P^T M P A P^T P x \\ &= x^T P^T B^T M B P x \\ &\leq x^T P^T M P x \\ &= Q_P(x), \end{aligned}$$

where the inequality follows by applying Eq. (III.1), which is satisfied by M , to the vector Px and the matrix B . We conclude that $Q_P \in \mathcal{Q}$.

Since the sum of matrices in \mathcal{Q} remains in \mathcal{Q} , it follows that $Z = \sum_{P \in \mathcal{P}} P^T M P$ belongs to \mathcal{Q} . ■

We define the “sample variance” $V(x)$ of the values x_1, \dots, x_n , by

$$V(x) = \sum_{i=1}^n (x_i - \bar{x})^2,$$

where $\bar{x} = (1/n) \sum_{i=1}^n x_i$. This is a nonnegative quadratic function of x , and therefore, $V(x) = x^T C x$, for a suitable nonnegative definite, nonzero symmetric matrix $C \in \mathbb{R}^{n \times n}$.

Lemma 7: There exists some $\alpha > 0$ such that

$$x^T Z x = \alpha V(x), \quad \text{for all } x \in \mathbb{R}^n.$$

Proof: We observe that the matrix Z satisfies

$$R^T Z R = Z, \quad \text{for all } R \in \mathcal{P}. \quad (\text{III.2})$$

To see this, fix R and notice that the mapping $P \mapsto PR$ is a bijection of \mathcal{P} onto itself, and therefore,

$$R^T Z R = \sum_{P \in \mathcal{P}} (PR)^T M (PR) = \sum_{P \in \mathcal{P}} P^T M P = Z.$$

We will now show that condition (III.2) determines Z , up to a multiplicative factor. Let z_{ij} be the (i, j) th entry of Z . Let $\mathbf{e}^{(i)}$ be the i th unit vector, so that $\mathbf{e}^{(i)T} Z \mathbf{e}^{(i)} = z_{ii}$. Let $P \in \mathcal{P}$ be a permutation matrix that satisfies $P\mathbf{e}^{(i)} = \mathbf{e}^{(j)}$. Then, $z_{ii} = \mathbf{e}^{(i)T} Z \mathbf{e}^{(i)} = \mathbf{e}^{(i)T} P^T Z P \mathbf{e}^{(i)} = \mathbf{e}^{(j)T} Z \mathbf{e}^{(j)} = z_{jj}$. Therefore, all diagonal entries of Z have a common value, to be denoted by z .

Let us now fix three distinct indices i, j, k , and let $y = \mathbf{e}^{(i)} + \mathbf{e}^{(j)}$, $w = \mathbf{e}^{(i)} + \mathbf{e}^{(k)}$. Let $P \in \mathcal{P}$ be a permutation matrix such that $P\mathbf{e}^{(i)} = \mathbf{e}^{(i)}$ and $P\mathbf{e}^{(j)} = \mathbf{e}^{(k)}$, so that $Py = w$. We have

$$2z + 2z_{ij} = y^T Z y = y^T P^T Z P y = w^T Z w = 2z + 2z_{ik}.$$

By repeating this argument for different choices of i, j, k , it follows that all off-diagonal entries of Z have a common value to be denoted by r . Using also the property that $Z\mathbf{e} = 0$, we obtain that $z + (n-1)r = 0$. This shows that the matrix Z is uniquely determined, up to a multiplicative factor.

We now observe that permuting the components of a vector x does not change the value of $V(x)$. Therefore, $V(x) = V(Px)$ for every $P \in \mathcal{P}$, which implies that $x^T P^T C P x = x^T C x$, for all $P \in \mathcal{P}$ and $x \in \mathbb{R}^n$. Thus, C satisfies (III.2). Since all matrices that satisfy (III.2) are scalar multiples of each other, the desired result follows. ■

Proof of Theorem 5: In view of Lemmas 6 and 7, if \mathcal{Q} is nonempty, then $V \in \mathcal{Q}$. Thus, it suffices to show that $V \notin \mathcal{Q}$. Suppose that $n \geq 8$, and consider the vector x with components $x_1 = 5$, $x_2 = x_3 = x_4 = 2$, $x_5 = 0$, $x_6 = x_7 = -3$, $x_8 = -5$, and $x_9 = \dots = x_n = 0$. We then have $V(x) = 80$. Consider the outcome of one iteration of the symmetric, equal-neighbor algorithm, if the graph has the form shown in Figure 1. After the iteration, we obtain the vector y with components $y_1 = 11/5$, $y_2 = y_3 = y_4 = 7/2$, $y_5 = 0$, $y_6 = y_7 = -4$, $y_8 = -11/4$, and

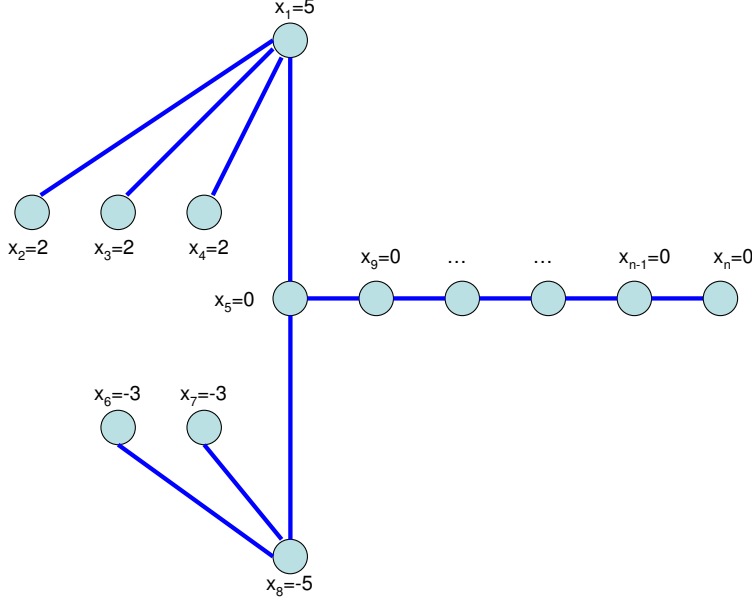


Fig. 1. A connected graph on n nodes showing that $V(x)$ is not a Lyapunov function when $n \geq 8$. All arcs of the form (i, i) are also assumed to be present, but are not shown. The nodes perform an iteration of the symmetric, equal-neighbor model according to this graph.

$y_9 = \dots = y_n = 0$. We have

$$\begin{aligned}
 V_n(y) &= \sum_{i=1}^n (y_i - \bar{y})^2 \\
 &\geq \sum_{i=1}^8 (y_i - \bar{y})^2 \\
 &\geq \sum_{i=1}^8 \left(y_i - \frac{1}{8} \sum_{i=1}^8 y_i \right)^2,
 \end{aligned} \tag{III.3}$$

where we used that $\sum_{i=1}^k (y_i - z)^2$ is minimized when $z = (1/k) \sum_{i=1}^k y_i$. A simple calculation shows that the expression (III.3) evaluates to $10246/127 \approx 80.68$, which implies that $V(y) > V(x)$. Thus, if $n \geq 8$, $V \notin \mathcal{Q}$, and the set \mathcal{Q} is empty. ■

IV. CONDITIONS FOR THE EXISTENCE OF A QUADRATIC LYAPUNOV FUNCTION

Are there some additional conditions (e.g., restricting the matrices A to a set smaller than \mathcal{A}), under which a quadratic Lyapunov function is guaranteed to exist? We start by showing

that the answer is positive for the case of a fixed matrix (that is, if the graph $G(t)$ is the same for all t).

Let A be a stochastic matrix, and suppose that there exists a positive vector π such that $\pi^T A = \pi^T$. Without loss of generality, we can assume that $\pi^T \mathbf{e} = 1$. It is known that in this case,

$$x^T A^T D A x \leq x^T D x, \quad \forall x \in \mathbb{R}^n, \quad (\text{IV.4})$$

where D is a diagonal matrix, whose i th diagonal entry is equal to π_i (cf. Lemma 6.4 in [2]). However, $x^T D x$ cannot be used as a Lyapunov function because $D\mathbf{e} \neq 0$ (cf. condition (c) in Definition 4). To remedy this, we argue as in [3] and define the matrix $H = I - \mathbf{e}\pi^T$, and consider the choice $M = H^T D H$. Note that M has rank $n - 1$.

We have $H\mathbf{e} = (I - \mathbf{e}\pi^T)\mathbf{e} = \mathbf{e} - \mathbf{e}(\pi^T \mathbf{e}) = \mathbf{e} - \mathbf{e} = 0$, as desired. Furthermore,

$$H A = A - \mathbf{e}\pi^T A = A - \mathbf{e}\pi^T = A - A\mathbf{e}\pi^T = A H.$$

Using this property, we obtain, for every $x \in \mathbb{R}^n$,

$$x^T A^T M A x = x^T A^T H^T D H A x = (x^T H^T) A^T D A (H x) \leq x^T H^T D H x = x^T M x,$$

where the inequality was obtained from (IV.4), applied to Hx . This shows that $H^T D H$ has the desired properties (a)-(c) of Definition 4, provided that \mathcal{A} is replaced with $\{A\}$.

We have just shown that every stochastic matrix (with a positive left eigenvector associated to the eigenvalue 1) is guaranteed to admit a quadratic Lyapunov function, in the sense of Definition 4. Moreover, our discussion implies that there are some classes of stochastic matrices \mathcal{B} for which the same Lyapunov function can be used for all matrices in the class.

- (a) Let \mathcal{B} be a set of stochastic matrices. Suppose that there exists a positive vector π such that $\pi^T \mathbf{e} = 1$, and $\pi^T A = \pi^T$ for all $A \in \mathcal{B}$. Then, there exists a nonzero, symmetric, nonnegative definite matrix M , of rank $n - 1$, such that $M\mathbf{e} = 0$, and $x^T A^T M A x \leq x^T M x$, for all x and $A \in \mathcal{B}$.
- (b) The condition in (a) above is automatically true if all the matrices in \mathcal{B} are doubly stochastic (recall that a matrix A is doubly stochastic if both A and A^T are stochastic); in that case, we can take $\pi = \mathbf{e}$.

- (c) The condition in (a) above holds if and only if there exists a positive vector π , such that $\pi^T Ax = \pi^T x$, for all $A \in \mathcal{B}$ and all x . In words, there must be a positive linear functional of the agents' opinions which is conserved at each iteration. For the case of doubly stochastic matrices, this linear functional is any positive multiple of the sum $\sum_{i=1}^n x_i$ of the agents' values (e.g., the average of these values).

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